The Open University

Science Supplementary Material
S382 Astrophysics/ S383 The Relativistic Universe

S382 / S383

Are you ready for S382 or S383?

Contents

If, after working through these notes, you are still unsure about whether S382 or S383 are the right courses for you, we advise you to seek further help and advice either from a Regional Advisor or from a Science Staff Tutor at your Regional Centre.

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1.1

Introduction

If you are intending to study S382 or S383 or both, you should make sure that you have the necessary background knowledge and skills to be able to enjoy either course fully and to give yourself the best possible chance of completing it successfully.

In order to help prepare you for studying S382 or S383 we have therefore produced two documents. The first is this *Are you ready for ...?* document, which includes a set of self-assessment questions designed for you to test your own preparedness. The second document is entitled *An Introduction to Astrophysics and Cosmology* comprising over 200 pages of revision material. This latter document is available for download from the Physics and Astronomy subject website

http://learn.open.ac.uk/site/physics-astronomy

by following the link to 'Extra study skills and resources'. It contains essentially all the mathematics, physics and astronomy that we expect you to be familiar with before embarking on S382 or S383.

First, you should read through these notes carefully and work through the self-assessment questions. This is a useful exercise for *all* prospective students of the course, even for those who have already studied other Open University science and mathematics courses and have completed the recommended prior study courses for S382 or S383 (see Section 1 below). Working through these diagnostic notes will serve as a reminder of some of the knowledge and skills which it is assumed that students will bring with them either from OU Level 2 science and mathematics courses or from other prior study.

If you find that you can answer virtually all of the questions in these notes, then it is likely that you are well prepared to take on S382 or S383. However, *if you find that you have substantial difficulties with more than two questions in any of Sections 2, 3, 4, 5 or 6*, then you should take one of two actions:

1. Ideally, you should consider taking one or more Level 2 OU courses that will prepare you for S382 or S383, as described in the following section. If you are not able to do this, or if you have already taken these courses but need reminding of their contents, then you should:
2. Spend time working through the relevant chapters of the document *An Introduction to Astrophysics and Cosmology*. The entire 200-page document is likely to require between 40 and 60 hours of study, if you need to study it all, but is not a substitute for taking the Level 2 courses themselves.

1 Suggested prior study

S382 *Astrophysics* and S383 *The Relativistic Universe* are Level 3 courses in astrophysics and cosmology which make intellectual demands appropriate to the third year of a conventional degree. Astrophysics and cosmology are subjects which rely on astronomy, physics, and most importantly mathematics. **In particular, you will be best prepared for these courses if you have previously studied the appropriate level of mathematics, which would include differential and integral calculus.**

So, before attempting either of S382 or S383, you are recommended to have good passes (pass 1 or 2) in Level 2 courses in astronomy, physics and mathematics. Suitable OU courses covering the astronomy background include S282 *Astronomy*, S283 *Planetary science and the search for life* and SXR208 *Observing the Universe* (or the predecessor course S281); for the physics background, S207 *The Physical World* (or its predecessor S271); and for the mathematics background MST209 *Mathematical Methods and Models* (or its predecessor MST207). These are by no means your only options. You will be able to start S382 or S383 if you have taken fairly recently, and passed well, courses equivalent to HND standard in physics or mathematics, or studied to at least the second year of a degree in one of these subjects. **If you are coming to S382 or S383 without having studied any of the Level 2 science or mathematics courses recommended above, it is essential that you establish whether or not your background and experience give you a sound basis on which to tackle the course.** The self-assessment questions in this document are designed to help you determine this.

2 Manipulating numbers and symbols

Mathematics is a vital tool in astrophysics and cosmology – it provides the language in which ideas are expressed and in which processes are described. Before embarking on either S382 or S383 you will therefore need to be fluent in your ability to manipulate and solve algebraic equations, including the use of powers, roots, reciprocals and imaginary numbers. You must be able to work with logarithmic ($\log_{10}$ and

$\log_e$) and trigonometric (sin, cos, tan) functions and understand what vectors and matrices represent and how they are combined. You should also be comfortable with graphical representations of equations, and be able to interpret what they show, as well as being able to calculate and combine measurement uncertainties.

If you have difficulty with more than two of these questions, you should consider taking an introductory mathematics course such as S151 or MST121. Alternatively you should work through Chapter 1 of *An Introduction to Astrophysics and Cosmology*.

Question 2.1 Simplify the following expression to the greatest possible extent.

$$\frac{(a^3)^{1/6} \times a^{-2}}{a^{-1/3}/a^{1/2}}.$$

Question 2.2 Combine the following two equations to eliminate m_2 and obtain an expression for v_2 in terms of q, i, P, m_1, G and π.

$$q = m_1/m_2$$

$$\frac{m_1^3 \sin^3 i}{(m_1 + m_2)^2} = \frac{P v_2^3}{2\pi G}.$$

Question 2.3 Find the values of x which satisfy the following quadratic equations, and then re-write each equation as the product of two factors. (a) $6x^2 - 7x - 153 = 0$, (b) $4x^2 + 81 = 0$.

Question 2.4 Assume $h = 6.626 \times 10^{-34}$ kg m^2 s^{-1}, $G = 6.67 \times 10^{-11}$ kg^{-1} m^3 s^{-2} and $c = 2.9979 \times 10^8$ m s^{-1}.

(a) Determine the SI unit of the quantity defined by

$$A = \sqrt{\frac{hc}{2\pi G}}.$$

(b) Calculate the numerical value of A and express your answer in scientific notation to an appropriate number of significant figures.

Question 2.5 The age t, spin period P and rate of change of spin period $\dot{P}$ of a pulsar are related by the equation $t = P/2\dot{P}$. Take logarithms of this equation and determine what is the gradient of lines of constant age on a graph of $\log \dot{P}$ versus $\log P$, and what is the vertical offset (interval) between lines of constant age for every factor of ten increase or decrease in age.

Question 2.6 Ten measurements are made of the wavelength of a spectral line in the spectrum of a star. The mean value of these measurements is 585 nm and their standard deviation is 6 nm.

(a) What uncertainty should be quoted for the mean wavelength? (b) Are the measurements consistent with a suspected true value for the wavelength of 591 nm? (c) If the mean value needed to be known with a precision of ± 1 nm, how many measurements of the wavelength would have to be made?

Question 2.7 A function $y(t)$ is believed to have the form $y(t) = at^n$, where a and n are unknown constants. Given a set of pairs of data y and t, what form of graph would you plot to enable you to determine the unknown constants?

Question 2.8 The shortest side of a right-angled triangle has a length of 5.0 cm and the smallest internal angle is equal to 0.395 radians. (a) What are the sizes of all the internal angles in degrees? (b) What are the lengths of the other two sides of the triangle?

Question 2.9 A three-dimensional position vector $\boldsymbol{r}$ has components (6.0 cm, 8.0 cm, 10.0 cm). (a) What are the components of the corresponding unit vector $\widehat{\boldsymbol{r}}$? (b) What is the result of multiplying a scalar by this unit vector? (c) What is the value of the scalar product $\boldsymbol{r} \cdot \boldsymbol{r}$?

Question 2.10 If matrices P and Q are as defined below, show that the transpose of the product of the two matrices $(PQ)'$ is equal to the product of the transposed matrices with the order reversed $Q'P'$.

$$P = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

$$Q = \begin{pmatrix} 3 & 5 \\ 7 & 9 \\ 11 & 13 \end{pmatrix}.$$

3 Stars and planets

Before embarking on S382, you should be familiar with some of the general terminology of astronomy and planetary science. You should also be aware, in general terms, of the structure and composition of stars and planets and of how stars form and evolve. You should be familiar with observational astronomy techniques, including how stars and exoplanets are observed, and the characteristics that quantify the performance of astronomical telescopes.

The material in this section is not as relevant for S383.

If you have difficulty with more than two of these questions, you should consider taking one or more introductory astronomy courses such as S282, S283 or SXR208. Alternatively you should work through Chapter 2 of *An Introduction to Astrophysics and Cosmology*.

Question 3.1 The star Regulus has a mass of 1.0×10^{31} kg, a radius of 2.45×10^{9} m and a luminosity of 1.7×10^{29} W. Express the mass, radius and luminosity of this star in solar units to an appropriate number of significant figures. (You may assume that $\mathrm{M}_{\odot} = 2.0 \times 10^{30}$ kg, $\mathrm{R}_{\odot} = 7.0 \times 10^{8}$ m and $\mathrm{L}_{\odot} = 3.8 \times 10^{26}$ W.)

Question 3.2 The star Altair has a trigonometric parallax of $\pi = 0.198''$, and a proper motion of $\mu = 0.66''$ per year. What are (a) its distance in units of pc (parsec), and (b) the magnitude of its transverse velocity in km s^{-1}? (Note that 1 pc $= 3.086 \times 10^{13}$ km.)

Question 3.3 Briefly outline the key events in the life cycle of a star and describe how the evolutionary track of a star may be represented on a Hertzsprung–Russell diagram.

Question 3.4 Explain how the absolute magnitude and spectral type of a star may be determined, in order to place it on the observational Hertzsprung–Russell diagram.

Question 3.5 Explain what is meant by the effective temperature of a star and state how it is related to the star's luminosity and radius.

Question 3.6 Use the classification of the following stars to arrange them in order of (a) increasing effective temperature, (b) increasing mass, (c) increasing radius, and (d) increasing luminosity.

Regulus (class B7 V), Procyon (class F5 V), Betelgeuse (class M2 I)

Question 3.7 What is the proton–proton chain?

Question 3.8 Broadly speaking, what are the two types of planet present in the Solar System, and how do their structural properties compare?

Question 3.9 (a) Briefly describe the three main techniques by which planets around other stars have been detected. (b) For a planet of a given mass, which methods are more likely to detect planets with small orbital radii rather than those with large orbital radii? (c) For a planet of a given orbital radius, which methods are more likely to detect planets with large masses or radii, rather than those with small masses or radii? (d) Aside from the obvious observational difficulties involving smaller brightness at greater distance, which methods work independently of distance?

Question 3.10 Consider the following characteristics of a visual telescope: (i) light-gathering power, (ii) field-of-view, (iii) angular magnification, (iv) limit of angular resolution. Summarise how each of these four characteristics depends on the aperture D_o and the focal length f_o of its objective lens or mirror (for a given eyepiece of focal length f_e).

4 Galaxies and the Universe

Before embarking on S383, you should be familiar with some of the general terminology of high-energy astronomy and cosmology. You should also be aware, in general terms, of the structure and evolution of galaxies, including active galaxies, and the structure and evolution of the Universe as a whole.

The material in this section is not as relevant for S382.

If you have difficulty with more than two of these questions, you should consider taking an introductory astronomy course such as S282. Alternatively you should work through Chapter 3 of *An Introduction to Astrophysics and Cosmology*.

Question 4.1 (a) Briefly outline the structure of our galaxy, the Milky Way, and describe its overall contents.

(b) State how other galaxies are classified according to their morphology.

Question 4.2 The radio galaxy 3C31 has a redshift of $z = 0.0169$.

(a) What is the apparent speed of recession of the galaxy in km s^{-1}?

(b) What is the distance to the galaxy? (Assume $H_0 = 70.4$ km s^{-1} Mpc^{-1}, $c = 3.00 \times 10^5$ km s^{-1}.)

Question 4.3 Define what is meant by the two forms of spectral flux density, F_λ and F_ν, and state how they are related.

Question 4.4 Summarise the unified model for AGN and describe how it explains the differences between type 1 Seyfert galaxies and type 2 Seyfert galaxies, and between radio galaxies and blazars.

Question 4.5 (a) Explain what is meant by the curvature of the Universe and describe how the scale factor $R(t)$ of the Universe varies with time in models for which $k = -1$, 0 or $+1$ with a cosmological constant $\Lambda = 0$. (b) How does the scale factor change with time in a universe with $k = 0$ and Λ positive?

Question 4.6 (a) Show that the Friedmann equation can be written as

$$H^2 = \frac{8\pi G}{3}\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - \frac{kc^2}{R^2},$$

where H is the Hubble parameter, R is the scale factor, ρ is the density of the Universe, k is the curvature parameter, Λ is the cosmological constant, c is the speed of light and G is the gravitational constant.

(b) Show that the equation above may itself be rewritten as

$$\Omega - 1 = \frac{kc^2}{R^2 H^2}$$

where $\Omega = \Omega_\mathrm{m} + \Omega_\Lambda$, i.e. the sum of the matter density and the density attributed to the cosmological constant, each expressed as a fraction of the critical density $\rho_\mathrm{crit} = 3H^2/8\pi G$.

(c) Hence justify the statement that if $\Omega_\mathrm{m} + \Omega_\Lambda = 1$, then $k = 0$.

Question 4.7 Draw a 'time-line' for the history of the Universe that indicates the major events that occurred at different times from the Planck time to the present day. Include on this time-line an indication of the temperature of the cosmic background radiation at the times of these events.

Question 4.8 (a) What are the current best estimates for the density parameters of the Universe ($\Omega_{\Lambda,0}$, $\Omega_{\mathrm{m},0}$ and $\Omega_{\mathrm{b},0}$), the curvature of the Universe k, the Hubble constant H_0, and the age of the Universe? (b) What are the two principal observational measurements which allow these determinations to be made?

5 Calculus

Before studying S382 or S383, in addition to the basic mathematics described earlier, you should also be comfortable with calculus representation (e.g. $\mathrm{d}x/\mathrm{d}t$ and $\int y\, \mathrm{d}x$) and be able to manipulate and solve differentials and integrals involving simple algebraic functions, exponentials and trigonometric functions. An awareness of methods such as the chain rule, product rule, integration by substitution and integration by parts will be a distinct advantage.

If you have difficulty with more than two of these exercises, you should consider taking a mathematics course such as MST209. Alternatively you should work through Chapter 4 of *An Introduction to Astrophysics and Cosmology*.

Question 5.1 A graph is plotted of the speed $v(t)$ of a particle as a function of time t. (a) What is signified by the gradient of the graph at a particular value of t and how may this be written as a differential function? (b) What is signified by the area under the graph between $t = t_1$ and $t = t_2$ and how may this be written as an integral function?

Question 5.2 If $y(t) = 6\sin(3t^2)$, use the chain rule to find $\mathrm{d}y/\mathrm{d}t$.

Question 5.3 If $y(x) = 2x^3/(x+3)^4$, use the product rule to find $\mathrm{d}y/\mathrm{d}x$.

Question 5.4 The pressure, volume and temperature in an interstellar gas cloud are related by $PV = NkT$, where N and k are constants. Use the technique of logarithmic differentiation with respect to time to find an expression for $\dot{T}/T$, where $\dot{T}$ is a shorthand for $\mathrm{d}T/\mathrm{d}t$.

Question 5.5 Approximate the function $f(x) = \exp(3x)$ using a second-order Maclaurin series expansion.

Question 5.6 If ρ is a scalar field describing the density inside a star, write down an expression for grad ρ and explain what it signifies.

Question 5.7 Evaluate the indefinite integral $\int \left(\frac{2}{x} + 3x^3\right)\,\mathrm{d}x$

Question 5.8 Evaluate the definite integral $\int_0^1 (x^2 - 1)^4\, 2x\,\mathrm{d}x$ by using the substitution $u = x^2 - 1$.

Question 5.9 Starting from the product rule for differentiation, derive the expression which describes the technique of integration by parts. In what circumstances can this technique make integration easier?

Question 5.10 If $\rho(r, \phi, \theta)$ is a function describing the density inside a star with reference to spherical polar coordinates, explain what is signified by the multiple integral

$$\int_{r=0}^{r=R} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \rho(r, \phi, \theta)\, r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

6 Physics

The physics subject-oriented knowledge required for S382 and S383 includes awareness of general concepts such as forces, work, energy, power and momentum, and familiarity with Newton's laws of motion and of gravity. You should recognize basic concepts in electricity and magnetism and properties of matter, particularly gases. You should also be comfortable with the idea of emission line, absorption line, and continuous spectra and the information which they convey. Finally, an awareness of general concepts in quantum physics, such as photons, energy levels and wave–particle duality will be useful, as will some familiarity with nuclear processes such as radioactivity, and a basic understanding of special relativity.

If you have difficulty with more than two of these questions, you should consider taking a physics course such as S207. Alternatively you should work through Chapter 5 of *An Introduction to Astrophysics and Cosmology*.

Question 6.1 A hydrogen molecule in an interstellar cloud of gas has a mass of 3.35×10^{-27} kg and a speed of 200 m s^{-1}. (a) What are its translational kinetic energy and the magnitude of its linear momentum? (b) If the molecule undergoes an acceleration of 5.00 m s^{-2} for 10.0 s, how much work is done on the molecule? (c) What is the magnitude of the force acting on the molecule whilst it is accelerating? (d) How far does the molecule travel whilst it is accelerating?

Question 6.2 Determine the mass of the Sun, given that the orbital radius of the planet Mercury is 5.79×10^{10} m and that its planetary year lasts 88.0 Earth days. ($G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.)

Question 6.3 A (super fast!) train of length 20 m in its own frame of reference (B) races towards a tunnel. In the frame of reference (A) in which the tunnel is at rest, the length of the tunnel is 10 m. The speed of the train is such that, in frame of reference A, the train length is contracted to 10 m.

(a) Calculate the speed of the train with respect to the tunnel.

(b) In frame of reference A the arrival of the front of the train at the far end of the tunnel (Event 1) is simultaneous with the rear end of the train entering the tunnel (Event 2) . Calculate the time interval between Event 1 and Event 2 in the train's reference frame (B).

Question 6.4 Particle A of rest mass m and translational kinetic energy $2mc^2$ hits and sticks to a stationary particle of rest mass $2m$. Use the equations of relativistic mechanics to find an expression for the rest mass M and the speed v_C of the composite particle C, both in terms of m only.

Question 6.5 The core of a star collapses suddenly to become a neutron star, and in the process no mass is lost. As a result of the collapse, what happens to each of the following quantities describing this system: (a) moment of inertia, (b) angular momentum, (c) angular speed, (d) rotational kinetic energy?

Question 6.6 Consider a fixed mass of an ideal gas. (a) What happens to the pressure exerted by the gas if it is allowed to expand whilst maintaining a constant temperature? (b) What happens to the pressure exerted by the gas if its temperature is increased whilst maintaining a constant volume of the gas? (c) How do the average energy and average speed of the gas molecules alter if the absolute temperature is doubled?

Question 6.7 (a) A hydrogen atom makes a transition from the $n = 5$ energy level to the $n = 1$ energy level. What is the energy of the photon that is emitted?

(b) A hydrogen atom absorbs a photon of energy 1.89 eV. Between which two energy levels does it make a transition?

(c) What happens if a hydrogen atom in its ground state absorbs a photon whose energy is 15.0 eV?

Question 6.8 The Boltzmann equation (below) predicts the number of atoms N_n in a particular energy level (with energy E_n) relative to the number of atoms N_1 in the ground state (with energy E_1) for a gas at a

temperature T. The constants g_n and g_1 are weighting factors and k is the Boltzmann constant. Describe briefly the effect on the population of energy levels as the temperature is increased.

$$\frac{N_n}{N_1} = \frac{g_n}{g_1} \exp\left(-\frac{(E_n - E_1)}{kT}\right).$$

(b) The Saha ionization equation (below) predicts the number of ions N^+ relative to the number of neutral atoms N_1 in the ground state for a sample of gas at a temperature T. The number of free electrons is given by N_e and I is the ionization energy. Other quantities are all constants. Describe briefly the effect on the relative numbers of ions and neutral atoms as the temperature is increased.

$$\frac{N^+}{N_1} = \left(\frac{2\pi m_\mathrm{e}}{h^2}\right)^{3/2} \frac{(kT)^{3/2}}{N_\mathrm{e}} \exp\left(-\frac{I}{kT}\right).$$

Question 6.9 The unstable isotope of carbon represented by ${}^{14}_{6}\mathrm{C}$ undergoes β^--decay. Write down a balanced equation to describe this process, indicating what nucleus is formed as a result.

Question 6.10 A particle of electric charge q travels in a region of uniform electric field of magnitude $\mathcal{E}$ and uniform magnetic field of magnitude B at a speed v. What are the magnitudes of the electric force and magnetic force acting on the particle and how do the directions in which these forces act differ?

Question 6.11 A beam of electromagnetic radiation has a frequency of 1.20×10^{20} Hz. (a) What is the wavelength of this radiation? (b) What is the energy, in electronvolts, of the photons of which the beam is composed? (c) For what temperature of black body spectrum would the mean photon energy have this value? (d) Which part of the electromagnetic spectrum corresponds to radiation of this photon energy? (You may assume $c = 3.00 \times 10^8$ m s^{-1}, $h = 6.63 \times 10^{-34}$ J s, $k = 1.38 \times 10^{-23}$ J K^{-1} and 1 eV $= 1.60 \times 10^{-19}$ J.)

Question 6.12 An interstellar gas cloud has an optical depth of $\tau = 10.0$ for radiation of a particular wavelength. By what percentage is the intensity of radiation of this wavelength reduced on passing through the cloud? Is the cloud optically thick or optically thin?

7 Other skills

Before studying S382 or S383 you will find it useful to have acquired the following skills:

Basic study skills: organize time for study, learn to pace study, read effectively to identify relevant information and data from scientific texts and accounts.

Writing skills: the ability to write coherently, present arguments in a logical sequence, and write a scientific account with appropriate diagrams.

S382 and S383 include significant use of a personal computer, including access to the Internet, as well as use of purpose-written computer-based interactive tutorials. Previous experience with using a computer would be helpful but is not essential.

8 References

In addition to the on-line document *An Introduction to Astrophysics and Cosmology*, which is specifically designed to aid you in your preparation for studying S382 and S383, you may find some of the following books useful.

The Sciences Good Study Guide, by A. Northedge, J. Thomas, A. Lane and A. Peasgood, Open University, 1997 (ISBN 0 7492 3411 3)

Basic Mathematics for the Physical Sciences, by R. Lambourne and M. Tinker, John Wiley and Sons, 2000 (ISBN 0 4718 5207 4).

Further Mathematics for the Physical Sciences, by M. Tinker and R. Lambourne, John Wiley and Sons, 2000 (ISBN 0 4718 6723 3).

Describing motion, by R. Lambourne and A. Durrant, The Open University, 2008, (ISBN 978 0 7492 1913 0)

Predicting motion, by R. Lambourne, The Open University, 2008,
(ISBN 978 0 7492 1914 7)

Classical physics of matter, by J. Bolton, The Open University, 2008, (ISBN 978 0 7492 1915 4)

Static fields and potentials, by J. Manners, The Open University, 2008, (ISBN 978 0 7492 1916 1)

Dynamic fields and waves, by A. Norton, The Open University, 2008, (ISBN 978 0 7492 1917 8)

Quantum physics: an introduction, by J. Manners, The Open University, 2008, (ISBN 978 0 7492 1918 5)

Quantum physics of matter, by A. Durrant, The Open University, 2008, (ISBN 978 0 7492 1919 2)

The fundamentals of physics, (6th ed.), by D. Halliday, R. Resnick and J. Walker, John Wiley and Sons, 2000, (ISBN 0 4713 9222 7)

Physics, by J. Cutnell and K. Johnson, John Wiley and Sons, 2000, (ISBN 0 4713 9219 7)

University Physics, by H. Benson, John Wiley and Sons, 1996, (ISBN 0 4710 0689 0)

Universe, (8th ed.), by Roger Freedman and William J. Kaufmann, W. H. Freeman, 2007, (ISBN 978 0 7167 8584 2)

An introduction to modern astrophysics, by B. W. Carroll and D. A. Ostlie, Addison Wesley Longman Publishing Co., 1996, (ISBN 0 2015 4730 9)

The dynamic Universe, (4th ed.), by T. P. Snow, West Publishing Company, 1991, (ISBN 0 3147 7104 2)

An introduction to the Sun and stars, by Simon F. Green and Mark H. Jones, Cambridge University Press, 2004, (ISBN 0 521 54622 2)

An introduction to galaxies and cosmology, by Mark H. Jones and Robert J. A. Lambourne, Cambridge University Press, 2004, (ISBN 0 521 54623 0)

An introduction to the Solar System, by Neil McBride and Iain Gilmour, Cambridge University Press, 2004, (ISBN 0 521 54620 6)

An introduction to astrobiology, by Iain Gilmour and Mark A. Sephton, Cambridge University Press, 2004, (ISBN 0 521 54621 4)

Observing the Universe, by Andrew J. Norton, Cambridge University Press, 2004, (ISBN 0 521 60393 5)

Answers

Answer 2.1

$$\frac{(a^3)^{1/6} \times a^{-2}}{a^{-1/3}/a^{1/2}} = \frac{a^{3/6} \times a^{1/3}}{a^2 \times a^{-1/2}} = \frac{a^{1/2} \times a^{1/3} \times a^{1/2}}{a^2}$$
$$= a^{(1/2)+(1/3)+(1/2)-2} = a^{(3+2+3-12)/6} = a^{-4/6} = a^{-2/3}.$$

Answer 2.2 From the first equation, $m_2 = m_1/q$, and substituting this into the second equation:

$$\frac{m_1^3 \sin^3 i}{\left(m_1 + \frac{m_1}{q}\right)^2} = \frac{Pv_2^3}{2\pi G}$$

Dividing both the top and bottom lines of the left-hand side of this by m_1^2 gives

$$\frac{m_1 \sin^3 i}{[1 + (1/q)]^2} = \frac{Pv_2^3}{2\pi G},$$

which may be rearranged to end up with

$$v_2 = \left(\frac{2\pi G m_1}{P[1 + (1/q)]^2}\right)^{1/3} \sin i.$$

Answer 2.3 (a) Using the general formula for solving a quadratic equation of the form $ax^2 + bx + c = 0$, namely $x = (-b \pm \sqrt{b^2 - 4ac}\,)/2a$, we have $x = (7 \pm \sqrt{49 - (4 \times 6 \times -153)}\,)/12 = (7 \pm \sqrt{3721}\,)/12$ $= (7 \pm 61)/12$. So $x = 5\frac{2}{3}$ or $x = -4\frac{1}{2}$. The original equation may therefore be written as $6(x - 5\frac{2}{3})(x + 4\frac{1}{2}) = 0$ or $(3x - 17)(2x + 9) = 0$.

(b) The equation may be rearranged as $4x^2 = -81$ or $2x = \mathrm{i}\sqrt{81}$, where i is the imaginary unit: $\mathrm{i} = \sqrt{-1}$. Hence the solutions are $x = \pm 9\mathrm{i}/2$ and the original equation may be rewritten as $4(x + 9\mathrm{i}/2)(x - 9\mathrm{i}/2) = 0$ or $(2x + 9\mathrm{i})(2x - 9\mathrm{i}) = 0$.

Answer 2.4 (a) Both the digit 2 and the constant π are dimensionless, i.e. they have no SI unit. So putting the units for h, c and G into the equation, the SI unit of A must be

$$\sqrt{\frac{\text{kg m}^2\,\text{s}^{-1} \times \text{m s}^{-1}}{\text{kg}^{-1}\,\text{m}^3\,\text{s}^{-2}}} = \sqrt{\frac{\text{kg m}^3\,\text{s}^{-2}}{\text{kg}^{-1}\,\text{m}^3\,\text{s}^{-2}}}$$

and then m^3 and s^{-2} cancel on the top and bottom lines, leaving

$$\sqrt{\frac{\text{kg}}{\text{kg}^{-1}}} = \sqrt{\text{kg}^2} = \text{kg}$$

and the SI unit of A is kg.

(b) Putting in the numbers:

$$A = \sqrt{\frac{6.626 \times 10^{-34} \times 2.9979 \times 10^8}{2\pi \times 6.67 \times 10^{-11}}}\ \text{kg} = 2.1771 \times 10^{-8}\ \text{kg}.$$

Since the values in the question are given to varying numbers of significant figures, the answer should be given to the same accuracy as the least accurate number given, which is G with only 3 significant figures. The final answer is therefore $A = 2.18 \times 10^{-8}$ kg.

Answer 2.5 Taking logarithms to the base 10 of the equation we obtain $\log t = \log P - \log 2 - \log \dot{P}$. This may be rearranged as $\log \dot{P} = \log P - \log t - \log 2$. So, on a graph of $\log \dot{P}$ versus $\log P$, lines of constant age will have a gradient of $+1$. For every factor of 10 increase (or decrease) in the age, the vertical offset (interval) between these lines will be -1 (or $+1$) logarithmic units respectively.

Answer 2.6 (a) The uncertainty σ_m in the mean value of n measurements is related to the standard deviation s_n of the measurements by $\sigma_m = s_n/\sqrt{n}$. So $\sigma_m = 6\ \text{nm}/\sqrt{10} \sim 2$ nm. Note that this is much smaller than the uncertainty in a single measurement, which is represented by the standard deviation of 6 nm.

(b) The difference between the mean value (585 nm) and the suspected true value (591 nm) is 6 nm, which is three times larger than the uncertainty in the mean. Assuming that values of the means that would be

obtained from many sets of ten measurements have a Gaussian distribution, then the probability of the value of the mean differing from the true value by three times the uncertainty in the mean is only 0.003. It is therefore unlikely, though possible, that the true value is 591 nm.

(c) If $\sigma_m = 1$ nm, and $s_n = 6$ nm, then $\sqrt{n} = s_n/\sigma_m = 6$, and so $n = 36$. So reducing the uncertainty from 2 nm to 1 nm would require almost four times as many measurements.

Answer 2.7 Taking logarithms (to the base 10) of each side of the proposed equation yields $\log y = \log a + n \log t$. So plotting a graph of $\log y$ against $\log t$ would yield a straight line whose intercept on the vertical axis is equal to $\log a$ and whose gradient is equal to n. Hence both a and n could be determined from the graph.

Answer 2.8 (a) An angle of 0.395 radians can be converted into degrees as follows. Since π radians is equal to $180°$, the angle in question is (0.395 radians) $\times(180°)/(\pi$ radians) $= 22.6°$. One of the other internal angles is a right angle (90°), so since the internal angles of a triangle add up to $180°$, the third angle is $180° - 90° - 22.6° = 67.4°$.

(b) Let the hypoteneuse of the triangle be of length h and the other unknown side be of length b. The smallest angle will be opposite the smallest side of the triangle, so $\sin 22.6° = 5.0$ cm$/h$ and therefore $h = 13.0$ cm. Similarly, $\tan 22.6° = 5.0$ cm$/b$, so $b = 12.0$ cm.

Answer 2.9 (a) A unit vector is defined as $\widehat{\boldsymbol{r}} = \boldsymbol{r}/r$, i.e. a vector $\boldsymbol{r}$ divided by its own magnitude $r = |\boldsymbol{r}|$; it is dimensionless. The magnitude of $\boldsymbol{r}$ is

$$r = |\boldsymbol{r}| = \sqrt{(6.0\text{ cm})^2 + (8.0\text{ cm})^2 + (10.0\text{ cm})^2} = 14.1\text{ cm}.$$

So the unit vector has components

$$\widehat{\boldsymbol{r}} = \left(\frac{6.0\text{ cm}}{14.1\text{ cm}}, \frac{8.0\text{ cm}}{14.1\text{ cm}}, \frac{10.0\text{ cm}}{14.1\text{ cm}}\right) = (0.43, 0.57, 0.71).$$

(b) The effect of multiplying a scalar by this unit vector is to produce a vector whose magnitude is equal to that of the original scalar, and whose direction is the same as that of the vector $\boldsymbol{r}$.

(c) The scalar product $\boldsymbol{r} \cdot \boldsymbol{r}$ is defined as $r^2 \cos\theta$ where, in this case, $\theta = 0°$ and so $\cos\theta = 1$. Alternatively $\boldsymbol{r} \cdot \boldsymbol{r} = (r_x^2 + r_y^2 + r_z^2)$. In either case the value of the scalar product is simply the magnitude of the vector $\boldsymbol{r}$ squared or

$$\boldsymbol{r} \cdot \boldsymbol{r} = |\boldsymbol{r}|^2 = (6.0\text{ cm})^2 + (8.0\text{ cm})^2 + (10.0\text{ cm})^2 = 200\text{ cm}^2.$$

Answer 2.10 The product of the two matrices is given by a 2×2 matrix whose elements are as follows:

$$PQ = \begin{pmatrix} (2\times 3) + (4\times 7) + (6\times 11) & (2\times 5) + (4\times 9) + (6\times 13) \\ (8\times 3) + (10\times 7) + (12\times 11) & (8\times 5) + (10\times 9) + (12\times 13) \end{pmatrix}.$$

Hence the resulting matrix is

$$PQ = \begin{pmatrix} 100 & 124 \\ 226 & 286 \end{pmatrix}.$$

The transpose of this product is simply

$$(PQ)' = \begin{pmatrix} 100 & 226 \\ 124 & 286 \end{pmatrix}.$$

Now, the transpose of the two original matrices are

$$P' = \begin{pmatrix} 2 & 8 \\ 4 & 10 \\ 6 & 12 \end{pmatrix}$$

$$Q' = \begin{pmatrix} 3 & 7 & 11 \\ 5 & 9 & 13 \end{pmatrix}$$

and the product of these with the order reversed is a 2×2 matrix with the following elements:

$$Q'P' = \begin{pmatrix} (3\times 2) + (7\times 4) + (11\times 6) & (3\times 8) + (7\times 10) + (11\times 12) \\ (5\times 2) + (9\times 4) + (13\times 6) & (5\times 8) + (9\times 10) + (13\times 12) \end{pmatrix}.$$

So the resulting matrix is

$$Q'P' = \begin{pmatrix} 100 & 226 \\ 124 & 286 \end{pmatrix}.$$

Hence $(PQ)' = Q'P'$ as required.

Answer 3.1 The mass of Regulus is $(1.0 \times 10^{31}\ \text{kg})/(2.0 \times 10^{30}\ \text{kg}\ \text{M}_\odot^{-1}) = 5.0\ \text{M}_\odot$.

The radius of Regulus is $(2.45 \times 10^{9}\ \text{m})/(7.0 \times 10^{8}\ \text{m}\ \text{R}_\odot^{-1}) = 3.5\ \text{R}_\odot$.

The luminosity of Regulus is $(1.7 \times 10^{29}\ \text{W})/(3.8 \times 10^{26}\ \text{W}\ \text{L}_\odot^{-1}) = 450\ \text{L}_\odot$.

All answers should be expressed to an accuracy of 2 s.f.

Answer 3.2 (a) Distance in parsec is related to trigonometric parallax by $d/\text{pc} = 1/(\pi/\text{arcsec})$. So in this case $d = 1/0.198 = 5.05$ pc.

(b) An angular shift of $0.66''$ at a distance of 5.05 pc corresponds to a displacement of $(5.05\ \text{pc}) \times \tan(0.66/3600)^\circ = 1.616 \times 10^{-5}$ pc. Converting this to a path length in kilometres yields $(1.616 \times 10^{-5}\ \text{pc}) \times (3.086 \times 10^{13}\ \text{km pc}^{-1}) = 4.987 \times 10^{8}$ km. This is the transverse distance travelled in one year, so the magnitude of the transverse velocity is $(4.987 \times 10^{8}$ km/year) / (365 days/year $\times$ 24 hours/day $\times$ 3600 seconds/hour) = 16 km s^{-1}.

Answer 3.3 Stars are formed by the collapse of fragments of dense molecular clouds. When the central regions become hot enough for nuclear fusion to be initiated, the star is born on the zero age main sequence of the H–R diagram. The star remains on the main sequence whilst undergoing hydrogen fusion in its core by the proton–proton chain or the CN cycle. When hydrogen in the core is exhausted, helium fusion may begin, and occurs by the triple alpha process. In low-mass stars this happens by way of an explosive helium flash. Other nuclear fusion reactions are subsequently possible in massive stars. When nuclear fuel is exhausted the star ends its life in one of several ways, depending on its mass. Low mass stars shed their outer layers as planetary nebulae and the core collapses to form a white dwarf. Massive stars explode as supernovae and the core collapses to form a neutron star or black hole.

A star's position on the H–R diagram is determined by its luminosity (vertical axis) and photospheric temperature (horizontal axis). For the majority of its life a star will remain in almost the same place on the H–R diagram, on the main sequence. As it undergoes different nuclear fusion reactions in its core, its luminosity and temperature will change, so it will move around the H–R diagram. The particular track followed depends essentially on the mass of the star.

Answer 3.4 A star's absolute magnitude M may be calculated by measuring its apparent magnitude m, and then correcting for its distance away d and the interstellar absorption which the light has suffered A. The relationship is $M = m + 5 - 5\log_{10} d + A$, where d is in parsec.

A star's spectral type is related to its photospheric temperature. A broad colour of the star may be determined as the difference between two photometric magnitudes, such as $m_\text{B} - m_\text{R}$; redder stars (with a larger value of $m_\text{B} - m_\text{R}$) are cooler whilst bluer stars (with a smaller value of $m_\text{B} - m_\text{R}$) are hotter. A more detailed measure of the temperature, and hence spectral type may be obtained from measuring the spectral lines from the star's photosphere. O-type stars have lines from ionized and neutral helium with weak hydrogen lines; B-type stars have weaker ionized helium lines, stronger neutral helium lines and stronger hydrogen lines; A-type stars have the strongest hydrogen lines and no helium lines; in F-type stars, hydrogen lines are weaker but ionized metal lines (calcium, iron, etc.) appear; whilst in G-type stars metal lines are stronger and hydrogen lines are very weak; K-type stars have the strongest calcium lines; the coolest M-type stars have molecular bands, notably those of titanium oxide.

Answer 3.5 The effective temperature of a star is the temperature of a black body source which has the same radius and luminosity as the star. The flux escaping through the star's surface is $F = L/4\pi R^2$, where L is the star's luminosity and R is the star's radius. However, the flux is related to the black body temperature by the Stefan–Boltzmann law: $F = \sigma T^4$, where σ is the Stefan–Boltzmann constant, 5.67×10^{-8} W m^{-2} K^{-4}. So the effective temperature of a star may be *defined* as $T_\text{eff} = \sqrt[4]{L/4\pi\sigma R^2}$.

Answer 3.6 From the classifications, Regulus is a relatively hot main sequence star, Procyon is a cooler main sequence star, and Betelgeuse is an even cooler supergiant star.

(a) In order of increasing effective temperature: Betelgeuse, Procyon, Regulus

(b) In order of increasing mass: Procyon, Regulus, Betelgeuse

(c) In order of increasing radius: Procyon, Regulus, Betelgeuse

(d) In order of increasing luminosity: Procyon, Regulus, Betelgeuse

In fact, the effective temperature, mass, radius and luminosity of the three stars (in solar units) are as shown in Table 1

Table 1 See Exercise 3.6

	effective temp. /K	mass /$M_\odot$	radius /$R_\odot$	luminosity /$L_\odot$
Betelgeuse (M2 I)	3500	20	800	50 000
Procyon (F5 V)	6500	1.3	1.2	2.5
Regulus (B7 V)	14 500	5.0	3.5	450

Answer 3.7 The proton–proton chain is the route by which low-mass stars fuse hydrogen nuclei (protons) to make nuclei of helium. The most common route for the p–p chain is as follows:

$$\begin{aligned} {}^1_1\mathrm{H} + {}^1_1\mathrm{H} &\rightarrow {}^2_1\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e} \\ {}^2_1\mathrm{H} + {}^1_1\mathrm{H} &\rightarrow {}^3_2\mathrm{He} + \gamma \\ {}^3_2\mathrm{He} + {}^3_2\mathrm{He} &\rightarrow {}^4_2\mathrm{He} + 2{}^1_1\mathrm{H}. \end{aligned}$$

About 25 MeV of energy is released for each helium nucleus which is formed in this way.

Answer 3.8 The Solar System is broadly separated into terrestrial planets (predominantly rocky and metallic in composition) in the inner region, and giant planets (predominantly gaseous) in the outer regions. The terrestrial planets have metallic cores overlaid by rocky mantles; some of them have atmospheres. The giant planets have small rocky cores, overlaid by either a thick layer of fluid helium and metallic hydrogen (Jupiter and Saturn) or liquid icy materials including water, ammonia and methane (Uranus and Neptune). Above this in each case is a thick atmosphere composed predominantly of hydrogen and helium.

Answer 3.9 (a) The three main techniques are as follows: (i) the Doppler spectroscopy or radial velocity technique whereby the periodic shift in spectral lines from a star indicates it is being tugged to-and-fro by an orbiting planet; (ii) the transit photometry or occultation technique whereby a periodic dip in the brightness of a star indicates the transit of an orbiting planet; (iii) the gravitational microlensing technique whereby the rapid brightening of a background star undergoing microlensing has a small spike superimposed on its lightcurve indicating the presence of a planet in the vicinity of the foreground star.

(b) A planet that is closer to its star will have a shorter orbital period. A planet with a small orbital radius is more likely to pass across the disc of the star, and will do so more often, than a planet with a larger orbital radius. Such a planet will also be more likely to give rise to a microlensing event which is close in time with the microlensing event due to its parent star. Close-in planets also have greater speeds and so can exhibit greater radial velocity Doppler shifts. Hence all three techniques are more likely to detect planets with small orbital radii rather than those with large orbital radii.

(c) Although the densities of planets do vary, it is fair to say that larger planets will generally also be more massive. Such large planets will block out more light from their parent star when they transit. Planets with large mass will have a larger Einstein ring and so will be more likely to microlens a background star, and will also have a longer microlensing timescale. A more massive planet will also cause a greater radial velocity Doppler shift, and so result in a more easily detectable wavelength shift. Hence all three techniques are likely to detect planets with large radii and masses rather than those with small radii and masses.

(d) Neglecting the fact that brightness will fall with distance, all three techniques are independent of the distance to the planetary system.

Answer 3.10 (i) The light-gathering power is proportional to D_o^2, but independent of f_o. (ii) The field-of-view is inversely proportional to f_o, and independent of D_o, although it does depend on the diameter of the eyepiece field stop. (iii) The angular magnification is proportional to f_o, but independent of D_o. (iv) The limit of angular resolution is inversely proportional to D_o and independent of f_o, but note that it varies with wavelength.

Answer 4.1 (a) The Milky Way is a typical spiral galaxy comprising a disc, a halo and a nuclear bulge. Young, high metallicity, population I stars are found mainly in the spiral arms, and somewhat older population I stars are found throughout the disc. Older, low metallicity, population II stars are found in the

halo (including globular clusters) and the nuclear bulge. HII regions (ionized hydrogen) and open star clusters are found in spiral arms and are associated with star formation.

(b) Other galaxies are classified according to their shape as elliptical, lenticular, spiral or irregular.

Answer 4.2 (a) For small redshifts, the speed of recession is related to the redshift by $v = zc$. So in this case $v = 0.0169 \times 3.00 \times 10^5$ km s^{-1} $= 5070$ km s^{-1}.

(b) The distance to the galaxy is related to its redshift by $d = cz/H_0$. So in this case

$$d = \frac{(3.00 \times 10^5 \text{ km s}^{-1}) \times 0.0169}{70.4 \text{ km s}^{-1} \text{ Mpc}^{-1}} = 72.0 \text{ Mpc}$$

Answer 4.3 Spectral flux density F_λ is the power per unit area per unit wavelength range received from an astronomical object. At a given wavelength, it is the rate at which energy from a source, with wavelengths in a narrow range $\Delta\lambda$, is delivered to a detector of area 1 m^2, divided by this wavelength range.

Similarly, spectral flux density F_ν is the power per unit area per unit frequency range received from an astronomical object. At a given frequency, it is the rate at which energy from a source, with frequencies in a narrow range $\Delta\nu$, is delivered to a detector of area 1 m^2, divided by this frequency range.

The two forms are related by $\nu\, F_\nu = \lambda\, F_\lambda$.

Answer 4.4 All AGN are believed to be powered by a central, supermassive black hole accreting via an accretion disc. Surrounding this is a torus of gas and dust. In radio-loud AGN, a pair of jets emerge perpendicular to the accretion disc. The region which produces the broad emission lines is close to the black hole, whilst the region that produces the narrow emission lines is further out. The unified model suggests that the main differences between subtypes of AGN are due to viewing the system at different inclination angles.

Seyfert galaxies are radio-quiet spiral galaxies: they have no jets. Type 2 Seyfert galaxies exhibit only narrow emission lines. They are believed to be oriented such that our view of the broad line region is obscured by the torus. Type 1 Seyfert galaxies exhibit both broad and narrow emission lines. They are believed to be oriented such that our view of the central engine is not obscured by the torus.

Radio galaxies and blazars are both radio-loud types of AGN. Radio galaxies are believed to be viewed from the side, such that we see one or both of the radio jets projected onto the plane of the sky. Blazars are believed to be viewed end-on, such that we are effectively looking down the axis of one of the jets.

Answer 4.5 (a) Curvature is a geometric property of spacetime that may be used to describe departures from flat geometry. If the Universe has a curvature parameter $k = 0$, then angles within a triangle add up to $180°$ and parallel lines remain parallel. If the Universe has a curvature parameter $k = +1$, then angles within a triangle add up to more than $180°$ and parallel lines may eventually converge. If the Universe has a curvature parameter $k = -1$, then angles within a triangle add up to less than $180°$ and parallel lines may eventually diverge.

In a universe with $k = -1$ and $\Lambda = 0$, the scale parameter will become ever larger as time progresses. Such a universe is said to be *open*. In a universe with $k = 0$ and $\Lambda = 0$, the scale parameter will become ever larger as time progresses, but the expansion will forever decelerate. Such a universe is said to be *critical*. Both the open and critical models correspond to a universe which is infinite at all times. In a universe with $k = +1$ and $\Lambda = 0$, the scale parameter will first increase and then decrease, as time progresses. Such a universe is said to be *closed* and is finite in size.

(b) In a universe with $k = 0$ and $\Lambda > 0$, the scale factor will forever increase, but the expansion will decelerate at early times, and later the deceleration will change into an acceleration.

Answer 4.6 (a) According to the Friedmann equation

$$\dot{R}^2 = \frac{8\pi G R^2}{3}\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - kc^2.$$

Now, $H(t) = \dot{R}(t)/R(t)$, so to get H^2 on the left-hand side of the Friedmann equation, divide both sides by R^2:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - \frac{kc^2}{R^2}$$

$$\text{i.e. } H^2 = \frac{8\pi G}{3}\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - \frac{kc^2}{R^2}$$

as required.

(b) Now multiplying both sides of this equation by $3/8\pi G$ gives

$$\frac{3H^2}{8\pi G} = \rho + \frac{\Lambda c^2}{8\pi G} - \frac{3kc^2}{8\pi G R^2}$$

$$\text{i.e. } \rho_{\text{crit}} = \rho + \rho_\Lambda - \frac{3H^2}{8\pi G}\frac{kc^2}{H^2R^2}$$

Dividing both sides by ρ_{crit} gives

$$1 = \frac{\rho}{\rho_{\text{crit}}} + \frac{\rho_\Lambda}{\rho_{\text{crit}}} - \frac{kc^2}{H^2R^2}$$

$$= \Omega_{\text{m}} + \Omega_\Lambda - \frac{kc^2}{H^2R^2}.$$

Letting $\Omega = \Omega_{\text{m}} + \Omega_\Lambda$, gives

$$\Omega - 1 = \frac{kc^2}{H^2R^2}$$

as required.

(c) Now, if $\Omega_{\text{m}} + \Omega_\Lambda = 1$, then $\Omega = 1$, so $\Omega - 1 = 0$. Therefore

$$\frac{kc^2}{H^2R^2} = 0$$

and hence $k = 0$ as required.

Answer 4.7 A time-line for the history of the Universe is shown in Figure 1.

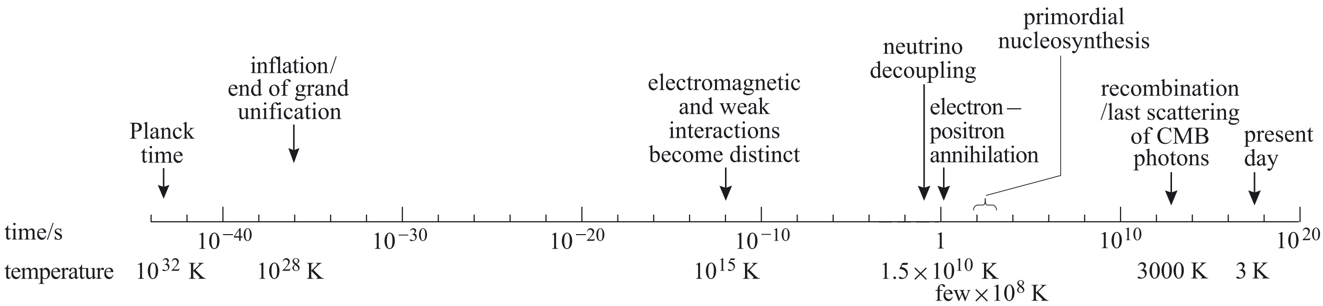

Figure 1 The major events in the evolution of the Universe from the Planck time to the present day.

Answer 4.8 (a) Recent results indicate $\Omega_{\Lambda,0} = 0.73$, $\Omega_{\text{m},0} = 0.27$ and $\Omega_{\text{b},0} = 0.044$, implying a Universe dominated by dark energy, and in which most of the matter is non-baryonic dark matter. The curvature of the Universe is likely $k = 0$ and the Hubble constant is measured to be $H_0 = 70.4$ km s^{-1} Mpc^{-1}. The corresponding age of the Universe is $t_0 = 13.7 \times 10^9$ years.

(b) The values above are principally based on measurements of the spatial fluctuations of the cosmic microwave background radiation provided by the Wilkinson Microwave Anisotropy Probe (WMAP) and measurements of the brightness of distant type Ia supernovae.

Answer 5.1 (a) The gradient of a graph of speed against time at a particular value of t is the magnitude of the acceleration (or the negative of the magnitude if the gradient is negative, i.e. a deceleration) of the particle at that instant of time. In symbols, $a(t) = \mathrm{d}v(t)/\mathrm{d}t$.

(b) The area under a graph of speed against time between two limits is the distance covered by the particle between these two times. In symbols $s(t) = \int_{t_1}^{t_2} v(t)\,\mathrm{d}t$.

Answer 5.2 Put $u = 3t^2$ then $\mathrm{d}u/\mathrm{d}t = 6t$. Also, $y = 6\sin u$ so $\mathrm{d}y/\mathrm{d}u = 6\cos u = 6\cos(3t^2)$.

Now, using the chain rule, $\mathrm{d}y/\mathrm{d}t = \mathrm{d}y/\mathrm{d}u \times \mathrm{d}u/\mathrm{d}t$, we have $\mathrm{d}y/\mathrm{d}t = 6\cos(3t^2) \times 6t = 36t\cos(3t^2)$.

Answer 5.3 First, put $u = 2x^3$ and $v = (x+3)^{-4}$, then $y = uv$ and we can use the product rule:

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

It is straightforward to find $\mathrm{d}u/\mathrm{d}x = 6x^2$, and $\mathrm{d}v/\mathrm{d}x = -4(x+3)^{-5}$. Finally putting this all together,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \left[2x^3 \times -4\,(x+3)^{-5}\right] + \left[(x+3)^{-4} \times 6x^2\right] \\ &= \frac{-8x^3}{(x+3)^5} + \frac{6x^2}{(x+3)^4}.\end{aligned}$$

Answer 5.4 Taking natural logarithms of each side of the equation,

$$\log_{\mathrm{e}} P + \log_{\mathrm{e}} V = \log_{\mathrm{e}} Nk + \log_{\mathrm{e}} T.$$

Now taking the time derivative of this expression, noting that in general $\mathrm{d}(\log_{\mathrm{e}} x)/\mathrm{d}t = \dot{x}/x$, and that N and k are constants, we have

$$\frac{\dot{P}}{P} + \frac{\dot{V}}{V} = \frac{\dot{T}}{T}.$$

Answer 5.5 The first and second derivatives of the function are $f'(x) = 3\exp(3x)$ and $f''(x) = 9\exp(3x)$. So the second-order Maclaurin series expansion is

$$\begin{aligned}\exp(3x) &= f(0) + xf'(0) + x^2 f''(0)/2! \\ \exp(3x) &= \exp 0 + x(3\exp 0) + x^2(9\exp 0)/2 \\ \exp(3x) &= 1 + 3x + 9x^2/2.\end{aligned}$$

Answer 5.6 In terms of components

$$\operatorname{grad}\rho = \nabla\rho = \left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}\right).$$

Grad ρ is a vector field which describes the density gradient within the star.

Answer 5.7

$$\int \left(\frac{2}{x} + 3x^3\right)\,\mathrm{d}x = 2\log_{\mathrm{e}} x + \frac{3x^4}{4} + C.$$

Answer 5.8 Setting $u = x^2 - 1$, clearly $\mathrm{d}u/\mathrm{d}x = 2x$ or $\mathrm{d}u = 2x\,\mathrm{d}x$. Substituting these into the original integral gives

$$\int_0^1 \left(x^2-1\right)^4 2x\,\mathrm{d}x = \int_{x=0}^{x=1} u^4\,\mathrm{d}u.$$

Now, evaluating this integral is straightforward:

$$\int_{x=0}^{x=1} u^4\,\mathrm{d}u = \left[\frac{u^5}{5}\right]_{x=0}^{x=1}.$$

We now reverse the original substitution to give

$$\left[\frac{(x^2-1)^5}{5}\right]_{x=0}^{x=1} = 0 - (-1)^5/5 = 1/5.$$

Answer 5.9 The product rule for differentiation is

$$\frac{\mathrm{d}(uv)}{\mathrm{d}t} = u\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}u}{\mathrm{d}t}$$

where u and v are both functions of t in this case. If we now *integrate* the above expression with respect to t we obtain

$$\int \frac{\mathrm{d}(uv)}{\mathrm{d}t}\,\mathrm{d}t = \int u\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t + \int v\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t$$

remembering that $\mathrm{d}t/\mathrm{d}t$ = 1, this simplifies to

$$uv = \int u\,\mathrm{d}v + \int v\,\mathrm{d}u.$$

Finally, rearranging this result we obtain the desired expression for integration by parts,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Success with the formula relies on choosing u and $\mathrm{d}v$ such that $\int v\,\mathrm{d}u$ is easier to calculate that $\int u\,\mathrm{d}v$.

Answer 5.10 This is a volume integral and the evaluation of it will give the total mass of the star.

Answer 6.1 (a) The translational kinetic energy of the molecule is

$$\begin{aligned} E_{\mathrm{KE}} &= 0.5mv^2 \\ &= 0.5 \times (3.35 \times 10^{-27}\ \mathrm{kg}) \times (200\ \mathrm{m\ s^{-1}})^2 \\ &= 6.70 \times 10^{-23}\ \mathrm{J}. \end{aligned}$$

The magnitude of the linear momentum of the molecule is

$$\begin{aligned} p &= mv \\ &= (3.35 \times 10^{-27}\ \mathrm{kg}) \times (200\ \mathrm{m\ s^{-1}}) \\ &= 6.70 \times 10^{-25}\ \mathrm{kg\ m\ s^{-1}}. \end{aligned}$$

(b) The work done on the molecule is equal to its change in kinetic energy (or equivalently the magnitude of the applied force multiplied by the distance over which it acts). The final speed of the molecule is found from

$$\begin{aligned} v &= u + at \\ &= (200\ \mathrm{m\ s^{-1}}) + (5.00\ \mathrm{m\ s^{-2}}) \times (10.0\ \mathrm{s}) \\ &= 250\ \mathrm{m\ s^{-1}}. \end{aligned}$$

The final translational kinetic energy is

$$\begin{aligned} E_{\mathrm{KE}} &= 0.5mv^2 \\ &= 0.5 \times (3.35 \times 10^{-27}\ \mathrm{kg}) \times (250\ \mathrm{m\ s^{-1}})^2 \\ &= 10.47 \times 10^{-23}\ \mathrm{J}. \end{aligned}$$

So the work done on the molecule is

$$\begin{aligned} W &= \Delta E_{\mathrm{KE}} \\ &= (10.47 - 6.70) \times 10^{-23}\ \mathrm{J} \\ &= 3.77 \times 10^{-23}\ \mathrm{J}. \end{aligned}$$

(c) Using a form of Newton's second law of motion, $F = ma$, the magnitude of the force acting on the molecule is

$$\begin{aligned} F &= (3.35 \times 10^{-27}\ \mathrm{kg}) \times (5.00\ \mathrm{m\ s^{-2}}) \\ &= 1.68 \times 10^{-26}\ \mathrm{N}. \end{aligned}$$

(d) The distance covered by the molecule whilst it is undergoing an acceleration may be found from

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= (200\ \mathrm{m\ s^{-1}} \times 10.0\ \mathrm{s}) + \left[\frac{1}{2} \times 5.00\ \mathrm{m\ s^{-2}} \times (10.0\ \mathrm{s})^2\right] \\ &= 2250\ \ \mathrm{m\ or\ \ 2.25\ km}. \end{aligned}$$

Answer 6.2 A planet of mass m moving in a circular orbit of radius r with uniform angular speed ω must be subject to a centripetal force of magnitude $F = mr\omega^2$. If this force is supplied by the gravitational attraction of the Sun, then by Newton's law of universal gravitation,

$$mr\omega^2 = \frac{GM_{\odot}m}{r^2}.$$

For a planet with angular speed ω, the orbital period is $P = 2\pi/\omega$, so replacing ω in the equation above by $2\pi/P$ gives

$$\frac{(2\pi)^2 mr}{P^2} = \frac{GM_\odot m}{r^2}.$$

Rearranging and cancelling the common terms, this yields

$$M_\odot = \frac{(2\pi)^2 r^3}{GP^2}.$$

Putting in the numbers we have,

$$\begin{aligned} M_\odot &= \frac{(2\pi)^2 \times (5.79 \times 10^{10}\,\mathrm{m})^3}{(6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}) \times (88.0 \times 24 \times 3600\,\mathrm{s})^2} \\ &= 1.99 \times 10^{30}\,\mathrm{kg}. \end{aligned}$$

Answer 6.3 (a) We use the formula for Lorentz contraction $L = L_0\sqrt{1 - V^2/c^2}$, where L is the length of the train in frame of reference A and L_0 is the length of the train in the frame of reference in which it is at rest, B. So 10 m = 20 m $\times\sqrt{1 - V^2/c^2}$. Hence $\sqrt{1 - V^2/c^2} = 1/2$ or $V^2/c^2 = 3/4$ and so $V = \sqrt{3}c/2$.

(b) In order to calculate the time interval between two events in the train's frame of reference, it is necessary to transform the time coordinates of Event 1 and Event 2 in frame of reference A to those in frame of reference B. Suppose that Event 1 and Event 2 occur at $(x_1 = 0,\, t_1 = 0)$ and $(x_2 = -L,\, t_2 = 0)$ respectively in frame A. We know from above that $V = \sqrt{3}c/2$ so $V^2 = 3c^2/4$ and therefore $1 - V^2/c^2 = 1/4$. Therefore, using the Lorentz transform,

$$t' = \frac{t - Vx/c^2}{\sqrt{1 - V^2/c^2}}$$

$$\text{Event 1:} \quad t_1' = \frac{t_1 - Vx/c^2}{\sqrt{1 - V^2/c^2}} = \frac{0 - 0}{1/2} = 0$$

$$\text{Event 2:} \quad t_2' = \frac{t_2 - Vx/c^2}{\sqrt{1 - V^2/c^2}} = \frac{0 + \sqrt{3}L/2c}{1/2} = \frac{\sqrt{3}L}{c}.$$

So $t_2' = (\sqrt{3} \times 10\ \mathrm{m})/(3.0 \times 10^8\ \mathrm{m\,s^{-1}}) = 5.8 \times 10^{-8}$ s. The time interval in frame of reference B is therefore such that Event 2 occurs 5.8×10^{-8} s after Event 1.

Answer 6.4 The following equations will be needed:

$$\begin{aligned} E_{\mathrm{TOT}} &= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \\ p &= \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \\ E_{\mathrm{TOT}}^2 &= p^2c^2 + m^2c^4. \end{aligned}$$

Now, let the total relativistic energy of particles A and B before the collision be $E_{\mathrm{A}} + E_{\mathrm{B}}$ and the total relativistic energy of particle C after the collision be E_{C}. Since relativistic energy is conserved, $E_{\mathrm{A}} + E_{\mathrm{B}} = E_{\mathrm{C}}$, but remember that each term can be made up of different amounts of relativistic translational kinetic energy and mass energy.

Similarly, let the magnitudes of the initial and final momenta be $p_{\mathrm{A}} + p_{\mathrm{B}}$ and p_{C} respectively. Since relativistic momentum is conserved $p_{\mathrm{A}} + p_{\mathrm{B}} = p_{\mathrm{C}}$.

Now, since the total relativistic energy is the sum of the relativistic translational kinetic energy and the mass energy, the initial energy of particle A is $E_{\mathrm{A}} = 2mc^2 + mc^2 = 3mc^2$, whereas the initial energy of particle B is $E_{\mathrm{B}} = 0 + 2mc^2 = 2mc^2$. Therefore $E_{\mathrm{A}} + E_{\mathrm{B}} = E_{\mathrm{C}} = 5mc^2$.

The initial momentum of particle A can be found from $E_{\mathrm{A}}^2 = p_{\mathrm{A}}^2c^2 + m^2c^4$, which in this case gives $(3mc^2)^2 = p_{\mathrm{A}}^2c^2 + m^2c^4$. Dividing through by c^2 and rearranging, we have $p_{\mathrm{A}}^2 = 8m^2c^2$, so $p_{\mathrm{A}} = \sqrt{8}mc$. Since particle B is stationary, $p_{\mathrm{B}} = 0$. Therefore $p_{\mathrm{A}} + p_{\mathrm{B}} = p_{\mathrm{C}} = \sqrt{8}mc$.

Now, the total relativistic energy of particle C is

$$E_{\mathrm{C}} = 5mc^2 = \frac{Mc^2}{\sqrt{1 - \frac{v_{\mathrm{C}}^2}{c^2}}}$$

and the relativistic momentum of particle C is

$$p_{\mathrm{C}} = \sqrt{8}mc = \frac{Mv_{\mathrm{C}}}{\sqrt{1 - \frac{v_{\mathrm{C}}^2}{c^2}}}.$$

To solve these two equations for M in terms of m, divide the energy equation through by c^2 and square the result to get

$$25m^2 = \frac{M^2c^2}{c^2 - v_{\mathrm{C}}^2}.$$

Rearranging further to make v_{C} the subject,

$$v_{\mathrm{C}}^2 = c^2 - \frac{M^2c^2}{25m^2}.$$

Now squaring the momentum equation gives

$$8m^2c^2 = \frac{M^2v_{\mathrm{C}}^2c^2}{c^2 - v_{\mathrm{C}}^2}.$$

Dividing this through by c^2 and rearranging we have

$$8m^2c^2 = v_{\mathrm{C}}^2\left(M^2 + 8m^2\right).$$

Now we have two equations for M and v_{C} in separate terms, and they can therefore be combined to find an expression for each in terms of m only. We have

$$\begin{aligned}
8m^2c^2 &= \left(c^2 - \frac{M^2c^2}{25m^2}\right)\left(M^2 + 8m^2\right) \\
8m^2c^2 &= M^2c^2 + 8m^2c^2 - \frac{M^4c^2}{25m^2} - \frac{8M^2c^2}{25} \\
\frac{M^4c^2}{25m^2} &= M^2c^2 - \frac{8M^2c^2}{25} \\
\frac{M^2}{25m^2} &= 1 - \frac{8}{25} \\
M^2 &= \frac{17}{25} \times 25m^2 \\
M^2 &= 17m^2
\end{aligned}$$

so the rest mass of the composite particle is $M = \sqrt{17}m$.

The speed of the particle is then

$$\begin{aligned}
v_{\mathrm{C}}^2 &= c^2 - \frac{M^2c^2}{25m^2} \\
v_{\mathrm{C}}^2 &= c^2 - \frac{17m^2c^2}{25m^2} \\
v_{\mathrm{C}}^2 &= c^2\left(1 - \frac{17}{25}\right) \\
v_{\mathrm{C}}^2 &= \frac{8c^2}{25}
\end{aligned}$$

so the speed of the composite particle is $v_{\mathrm{C}} = \sqrt{8}c/5$, which is about 57% of the speed of light.

Answer 6.5 (a) The moment of inertia I of a solid sphere is proportional to its radius squared (in fact for a uniform sphere $I = 2MR^2/5$). So, since the mass M remains constant but the radius R is reduced, the moment of inertia will *decrease*.

(b) Since no external torques act, the angular momentum of the system is *constant* ($\Gamma = \mathrm{d}J/\mathrm{d}t = 0$).

(c) Since the magnitude of the angular momentum J depends on the product of the moment of inertia and ω, the angular speed ($J = I\omega$), the angular speed must *increase* to compensate for the decrease in the moment of inertia.

(d) The rotational kinetic energy depends on the moment of inertia and the square of the angular speed ($E_{\mathrm{rot}} = \frac{1}{2}I\omega^2$), but $I\omega$ is constant and ω increases, so the rotational kinetic energy must also *increase*.

Answer 6.6 (a) Using the ideal gas equation, $PV = NkT$, if V increases whilst N and T are kept fixed, then clearly P must decrease. So the pressure exerted by the gas will decrease.

(b) Using the ideal gas equation again, if T increases whilst N and V are kept fixed, then clearly P must increase. So the pressure exerted by the gas will increase.

(c) The average energy of the molecules is proportional to the absolute temperature and the average speed of the molecules is proportional to the square root of the absolute temperature. So if the temperature is doubled, the average energy also doubles, whereas the average speed increases by a factor of $\sqrt{2}$.

Answer 6.7 The energy levels of the hydrogen atom are determined by the equation $E_n = -13.60\ \mathrm{eV}/n^2$, where n is an integer. The energies of the first six energy levels are therefore: $E_1 = -13.60$ eV, $E_2 = -3.40$ eV, $E_3 = -1.51$ eV, $E_4 = -0.85$ eV, $E_5 = -0.54$ eV and $E_6 = -0.38$ eV.

(a) In making a transition from the $n = 5$ energy level to the $n = 1$ energy level, the atom emits a photon of energy (–0.54 eV) – (–13.60 eV) = 13.06 eV.

(b) In order to absorb a photon of energy 1.89 eV, a hydrogen atom must make a transition *from* the $n = 2$ energy level *to* the $n = 3$ energy level.

(c) The atom is ionized. The first 13.6 eV is used to raise the atom from the ground-state energy level to a state in which the nucleus (proton) and electron are widely separated. The remaining 1.4 eV is transferred to the proton and electron as kinetic energy.

Answer 6.8 (a) The Boltzmann equation predicts that, as the temperature increases, so the number of atoms in higher energy levels relative to the number in the ground state also increases.

(b) The Saha ionization equation predicts that, as the temperature increases, so the number of ions relative to the number of neutral atoms in the ground state also increases.

Answer 6.9 A β^--decay occurs when a neutron transforms into a proton. So the mass number of the nucleus remains the same, but its atomic number increases by one. The balanced equation is

$$^{14}_{6}\mathrm{C} \rightarrow\ ^{14}_{7}\mathrm{N} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}.$$

The resulting nucleus is an isotope of nitrogen and an electron (β-particle) and an electron antineutrino are emitted, carrying away the released energy.

Answer 6.10 The magnitude of the electric force is $F_{\mathrm{el}} = q\mathcal{E}$ and the magnitude of the magnetic force is $F_{\mathrm{mag}} = qvB\sin\theta$, where θ is the angle between the velocity vector of the particle and the direction of the magnetic field. The direction of the electric force vector is parallel to the direction of the electric field, whilst the magnetic force acts in a direction which is at *right angles* to the plane containing the velocity vector of the particle and the magnetic field vector.

Answer 6.11 (a) Using $c = \lambda\nu$, the wavelength of the radiation is

$$\begin{aligned}\lambda &= (3.00 \times 10^8\ \mathrm{m\ s^{-1}})/(1.20 \times 10^{20}\ \mathrm{Hz})\\ &= 2.50 \times 10^{-12}\ \ \mathrm{m\ or\ \ 0.00250\ nm}.\end{aligned}$$

(b) Using $E_{\mathrm{ph}} = h\nu$, the energy of the photons of which this radiation is composed is

$$\begin{aligned}E_{\mathrm{ph}} &= (6.63 \times 10^{-34}\ \mathrm{J\ s}) \times (1.20 \times 10^{20}\ \mathrm{Hz})\\ &= 7.96 \times 10^{-14}\ \mathrm{J}.\end{aligned}$$

Converting this into electronvolts:

$$\begin{aligned}E_{\mathrm{ph}} &= (7.96 \times 10^{-14}\ \mathrm{J})/(1.60 \times 10^{-19}\ \mathrm{J\ eV^{-1}})\\ &= 4.98 \times 10^5\ \ \mathrm{eV\ or\ about\ 500\ keV}.\end{aligned}$$

(c) Using $\langle E_{\text{ph}} \rangle = 2.70kT$, a black body spectrum whose mean photon energy is 500 keV would have a temperature of

$$\begin{aligned} T &= \langle E_{\text{ph}} \rangle / 2.70k \\ &= (7.96 \times 10^{-14}\ \text{J})/(2.70 \times 1.38 \times 10^{-23}\ \text{J K}^{-1}) \\ &= 2.14 \times 10^{9}\ \text{K} \end{aligned}$$

or just over 2 billion kelvin.

(d) Photons with this energy lie close to the boundary between the X-ray and gamma-ray parts of the electromagnetic spectrum.

Answer 6.12 The ratio of the amount of incident radiation to the amount transmitted is

$$\frac{I_x}{I_0} = \exp(-\tau) = \exp(-10.0) = 4.54 \times 10^{-5}.$$

So the intensity of the radiation is reduced by $(1 - 4.54 \times 10^{-5}) \times 100\% = 99.995\%.$ The cloud is optically thick.